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# Generalized supremum in ordered linear space and facial structure of a convex set

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## §1 DEFINITIONS AND BASIC RESULTS

Let  $E$  be a linear space over  $\mathbb{R}$ , and  $P$  be a convex cone in  $E$  satisfying

$$(P1) \quad E = P - P,$$

$$(P2) \quad P \cap (-P) = \{0\}.$$

An order relation in  $E$  can be defined by  $x \leq y \iff y - x \in P$ . It can easily be seen that

$$(1) \quad x \leq y \text{ and } y \leq x \implies x = y,$$

$$(2) \quad x \leq y \text{ and } y \leq z \implies x \leq z,$$

$$(3) \quad x \leq y \implies x + z \leq y + z \text{ for all } z \in E,$$

$$(4) \quad 0 \leq x \text{ and } 0 \leq \lambda \in \mathbb{R} \implies 0 \leq \lambda x,$$

$$(5) \quad \text{For every } x \in E, \text{ there exists } x_1, x_2 \in E \text{ such that } x = x_1 - x_2, \text{ and } 0 \leq x_1, x_2.$$

Conversely, if an order in  $E$  satisfies (1)  $\sim$  (5), then  $P = \{x \in E \mid 0 \leq x\}$  is a convex cone satisfying (P1) and (P2). A linear space  $E$  equipped with such a positive cone  $P$  is called a partially ordered linear space, and is sometimes denoted by  $(E, P)$ .

**Definition.** For a subset  $A$  of  $E$ , the generalized supremum  $\text{Sup } A$  is defined to be the set of all minimal elements of  $U(A)$ , where  $U(A)$  is the set of all upper bound of  $A$ .

We say in other words that  $a \in \text{Sup } A$  if and only if  $a \leq b$  whenever  $b \in U(A)$  and  $a, b$  are comparable. The generalized infimum  $\text{Inf } A$  can be defined similarly. In order to distinguish this notion from the least upper bound and the greatest lower bound, we denote the latter ones by  $\sup A$  and  $\inf A$  respectively. If  $E$  is order complete, then  $\text{Sup } A = \{\sup A\}$  holds whenever the subset  $A$  is upper bounded (i.e.,  $U(A) \neq \emptyset$ ). When  $E = \mathbb{R}^n$  and  $P$  is closed and not a lattice cone,  $\text{Sup } A$  becomes a infinite set in most cases. However, it is possibly empty, even when  $A$  is upper bounded.

**Proposition 1.** For  $a \in E$  and  $\lambda > 0$ , we have

$$(1) \quad \text{Sup}(A + a) = \text{Sup } A + a,$$

$$(2) \quad \text{Sup } \lambda A = \lambda \text{Sup } A,$$

$$(3) \quad \text{Sup } A = -\text{Inf}(-A).$$

**Proposition 2.** For an arbitrary set  $A \subset E$  with  $U(A) \neq \emptyset$ ,

$$\text{Sup } A = \text{Sup}(coA)$$

holds where  $coA$  is the convex hull of  $A$ .

*proof.* It suffices to show that  $U(A) = U(coA)$ . Take  $x_0 \in U(A)$  arbitrarily. For  $x \in coA$  there exist some points  $x_1, x_2, \dots, x_n$  in  $A$  such that  $x = \sum_{i=1}^n \lambda_i x_i$  with  $0 \leq \lambda_i \leq 1$  and  $\sum_{i=1}^n \lambda_i = 1$ . Hence  $x_0 - x = \sum_{i=1}^n \lambda_i (x_0 - x_i) \geq 0$  and we have  $x_0 \in U(coA)$ .

When  $A$  is a finite set of the form  $\{a_1, \dots, a_n\}$ , we denote the set of the upper bound of  $A$  by  $U(a_1, \dots, a_n)$  instead of  $U(A)$ . With this notation, we define  $a \vee b$  ( $a, b \in E$ ) to be the set of all minimal elements of  $U(a, b)$ . Also  $a \wedge b$  can be defined similarly. When  $(E, P)$  is a lattice,  $a \vee b$  is always a single element which is the minimum of  $U(a, b)$ .

**Proposition 3.** For every  $a, b, c \in E$  and  $\lambda \in \mathbb{R}$ ,

- (1)  $(a + c) \vee (b + c) = (a \vee b) + c$ ,
- (2)  $\lambda a \vee \lambda b = \lambda(a \vee b)$ .

**Theorem 1.** For  $a, b \in E$ ,  $a \vee b \neq \emptyset$  implies  $a \wedge b \neq \emptyset$  and the converse is also true. Moreover,

$$a + b - (a \vee b) = a \wedge b$$

holds and in particular we have  $a \in a_+ + a_-$  where  $a_+ = a \vee 0$  and  $a_- = a \wedge 0$ .

The proof of Theorem 1 can be seen in [ 6 ]. Also, some examples in which  $a \vee b$  can be empty are shown.

A partially ordered linear space  $(E, P)$  is said to be monotone order complete (m.o.c. for short) if every upper bounded totally ordered subset of  $E$  has the least upper bound in  $E$ . The followings are known.

**Proposition 4.** In the case  $E = \mathbb{R}^d$ ,  $(E, P)$  is m.o.c. if and only if  $P$  is closed.

**Proposition 5.** Suppose that  $E$  is a Banach space and  $P$  is closed. Let  $E^*$  be the topological dual of  $E$  and let  $P^* = \{x^* \in E^* \mid x^*(x) \geq 0, x \in P\}$ . If  $P^* - P^* = E^*$ , then  $(E^*, P^*)$  is m.o.c.

The proof can be done by using Banach Steinhaus theorem, and in [ 2 ], one can see some conditions under which  $P^* - P^* = E^*$  holds.

A linear topology of  $(E, P)$  is called an order continuous topology if every decreasing net  $\{a_\lambda\} \subset E$  with  $\inf a_\lambda = 0$  converges to 0 by the topology. We consider some further conditions for  $P$ ;

(P3)  $P$  is closed with respect to an order continuous topology,

(P4) For every decreasing net  $\{a_\lambda\}$  in  $P$ ,  $\inf a_\lambda = a$  implies  $a \in P$ .

Note that (P3) implies (P4).

**Theorem 2.** Suppose that a partially ordered linear space  $(E, P)$  is monotone order complete and  $P$  satisfies (P3) or (P4). Then for every subset  $A$  of  $E$ ,

$$U(A) = (\text{Sup } A) + P$$

holds. In particular,  $a \vee b \neq \emptyset, a \wedge b \neq \emptyset$  for every  $a, b \in E$ , and  $U(a, b) = (a \vee b) + P$ .

*proof.* It suffices to show that  $U(A) \subset (\text{Sup } A) + P$ . For an arbitrary  $x \in U(A)$ , the section  $U(A)_x = \{y \in U(A) \mid y \leq x\}$  is a nonempty convex set in  $E$ . If  $T \subset U(A)_x$  is a totally ordered subset, then by monotone order completeness, there exists a greatest lower bound  $z_0$  of  $T$ . Since  $T \subset U(A) = \bigcap_{y \in A} (y + P)$ , (P4) yields  $z_0 \in U(A)_x$ . Hence by Zorn's lemma,  $U(A)_x$  has at least a minimal element  $y_0$ . It is easy to see that  $y_0$  is also a minimal element of  $U(A)$ , and it means that  $x \in (\text{Sup } A) + P$ . The second statement of the theorem is obvious. Indeed,  $U(a, b)$  is always nonempty because  $P - P = E$ . Hence it is sufficient to use the first statement. Q.E.D.

**Corollary 1.** Suppose that  $(E, P)$  satisfies the hypotheses in Theorem 2 and let  $A$  be a subset of  $E$ . If  $\text{Sup } A$  consists of a single element  $a$ , then  $a$  is the least upper bound of  $A$ .

**Corollary 2.** *For every subset  $A$  of  $E$ ,  $U(L(U(A))) = U(A)$  holds where  $L(U(A))$  denotes the lower bound of  $U(A)$ . Moreover, if  $(E, P)$  satisfies the hypotheses in Theorem 2, then we have  $\text{Sup Inf Sup } A = \text{Sup } A$ .*

Next we give another sufficient condition for the same results by considering the faces of  $P$ . Moreover, we will give an example which shows that each of the two conditions does not imply the other.

## §2 FACES OF THE POSITIVE CONE

Let  $(E, P)$  be a partially ordered linear space, and suppose that  $P$  is algebraically closed, that is, every straight line of  $E$  meets  $P$  by a closed interval. A point  $x$  of a convex subset  $A \subset E$  is called an algebraic interior point of  $A$  if for every  $z \in E$ , there exists  $\lambda > 0$  such that  $x + \lambda z \in A$ . Algebraic exterior points are defined similarly, and we denote the algebraic interior (exterior) of  $A$  by  $\text{int}A$  ( $\text{ext}A$ ) respectively. Moreover,  $\partial A = (\text{int}A \cup \text{ext}A)^c$  is called the algebraic boundary of  $A$ . A convex subset  $C$  of  $P$  is called an exposed face of  $P$  if there exists a supporting hyperplane  $H$  of  $P$  such that  $C = P \cap H$ . By  $\mathfrak{F}(P)$ , we denote the set of all exposed faces of  $P$ . For  $C \in \mathfrak{F}(P)$ ,  $\dim C$  is defined as the dimension of  $\text{aff}C$  where  $\text{aff}C$  denotes the affine hull of  $C$ . The following theorem is a fundamental result, and is also useful when we intend to determine the set  $a \vee b$  explicitly.

**Theorem 3.** *Suppose that  $P$  is algebraically closed and  $\text{int } P \neq \emptyset$ . If  $\dim C \leq 1$  for every  $C \in \mathfrak{F}(P)$ , then*

$$a \vee b = \partial U(a) \cap \partial U(b)$$

*holds for every incomparable pair  $a, b \in E$ .*

In the case when a linear topology is given in  $E$ , the assertion of Theorem 3 can be translated into the terms of topology and is still valid.

**Lemma 1.** *If  $0 \leq x \leq y$  and  $y \in \partial P$ , then  $x \in \partial P$ .*

*proof.* Suppose that  $x \in \text{int } P$  and put  $z = 2y - x$ , then  $z = y + (y - x) \in P + P = P$ . Since  $P$  is convex and  $x \in \text{int } P$ ,  $y = \frac{1}{2}(x + z) \in \text{int } P$ . This contradicts the assumption.

*proof of Theorem 3.* Let  $x_0$  be an element of  $a \vee b$ , and suppose that  $x_0 \in \text{int } U(a)$ . Then there exists  $\lambda > 0$  such that  $c \stackrel{\text{def}}{=} (1 - \lambda)x_0 + \lambda b \in U(a)$ . It is easy to see that  $c \in U(a) \cap U(b) = U(a, b)$  and  $c \not\leq x_0$ . This contradicts the fact that  $x_0$  is a minimal element of  $U(a, b)$ , and hence  $a \vee b \subset \partial U(a) \cap \partial U(b)$ .

Conversely, take  $x_0 \in \partial U(a) \cap \partial U(b)$  arbitrarily and suppose that  $y_0 \leq x_0$ ,  $y_0 \in U(a, b)$ . Since  $a \leq y_0 \leq x_0$ , it follows by Lemma 1 that

$$y_0 \in [a, x_0] \subset \partial U(a),$$

where  $[a, x_0] = \{x \in E \mid a \leq x \leq x_0\}$  is an order interval. Obviously every order interval is a convex set. Similarly we have

$$y_0 \in [b, x_0] \subset \partial U(b),$$

and hence

$$[a, x_0] \cap \text{int } U(a) = \emptyset, \quad [b, x_0] \cap \text{int } U(b) = \emptyset,$$

while  $\text{int } U(a)$  and  $\text{int } U(b)$  are both assumed to be nonempty. Applying the separation theorem, we can find hyperplanes  $H_1, H_2$  of  $E$  such that

- (1)  $H_1$  separates  $[a, x_0]$  and  $U(a)$  and,
- (2)  $H_2$  separates  $[b, x_0]$  and  $U(b)$ .

Since  $[a, x_0] \subset U(a)$  and  $[b, x_0] \subset U(b)$ , we can see that  $[a, x_0] \subset U(a) \cap H_1$  and  $[b, x_0] \subset U(b) \cap H_2$ . By the condition of  $\mathfrak{F}(P)$ , these two faces are actually half lines. On the other hand,  $a, b$ , and  $x_0$  cannot be in any single straight line because  $a$  and  $b$  are not comparable. Hence  $[a, x_0]$  and  $[b, x_0]$  are respectively included in two different lines, and in particular, both  $x_0$  and  $y_0$  belong to the intersection of those two lines. This means  $x_0 = y_0$  and so  $x_0 \in a \vee b$ . Q.E.D.

**Lemma 2.** *Suppose that the positive cone  $P$  is algebraically closed and  $\text{int } P \neq \emptyset$ . Then  $\partial U(a) \cap \partial U(b) \neq \emptyset$  for every incomparable pair  $a, b \in E$ .*

*proof.* We can take an element  $x \in U(a) \cap U(b)$ . Indeed,  $b - a$  can be written in the form  $p - q$  with  $p, q \in P$ , and so  $a + p = b + q \in U(a) \cap U(b)$ . Since  $a \notin U(b)$ , and  $U(b)$  is algebraically closed, there exists  $\lambda_0 \in [0, 1)$  such that  $\lambda_0 = \max\{\lambda > 0 \mid x + \lambda(a - x) \in U(b)\}$ . Obviously,  $z_0 \stackrel{\text{def}}{=} x + \lambda_0(a - x) \in U(a) \cap \partial U(b)$ . Next we take  $\lambda_1 = \max\{\lambda \mid z_0 + \lambda(b - z_0) \in U(a)\}$  similarly. Then  $z_1 \stackrel{\text{def}}{=} z_0 + \lambda_1(b - z_0) \in \partial U(a)$ . Moreover, since  $b \leq z_1 \leq z_0 \in \partial U(b)$ , it follows by Lemma 1 that  $z_1 \in \partial U(b)$ .

Applying Theorem 3 and Lemma 2, we can obtain the following.

**Corollary 3.** *Under the hypotheses in Theorem 3,  $a \vee b \neq \emptyset$  holds for every  $a, b \in E$ . Moreover when  $a$  and  $b$  are not comparable, we have*

$$U(a, b) = (a \vee b) + P.$$

*proof.* The first statement of the theorem follows immediately from Theorem 3 and Lemma 2. To see the latter, it is sufficient to show  $U(a, b) \subset (a \vee b) + P$ . For an arbitrary element  $x \in U(a, b)$ , we can choose  $z_1$  as in the proof of Lemma 2. Then  $z_1 \leq x$  and  $z_1 \in \partial U(a) \cap \partial U(b)$ . Hence by Theorem 3,  $z_1 \in a \vee b$ , and this means that  $x \in (a \vee b) + P$ .

**Theorem 4.** *Under the hypotheses in Theorem 3,*

$$U(A) = (\text{Sup } A) + P$$

*holds for every subset  $A \subset E$ . In particular, the conclusions in Corollary 1 and Corollary 2 are valid.*

*Remark.* The hypotheses of this theorem can be somewhat weakened. Moreover, using this theorem, we can simplify the proof of Lemma 2 and can obtain the second statement of Corollary 3 directly.

**Lemma 3.** *If  $x \in \partial U(A)$  for a subset  $A$  of  $E$ , then  $U(A)_x \subset \partial U(A)$  where  $U(A)_x = \{y \in U(A) \mid y \leq x\}$ .*

*proof.* Let  $y$  be an arbitrary point in  $U(A)_x$ . Since  $x \in \partial U(A)$  there exists a point  $z \in E$  such that  $\{x + tz \mid t > 0\} \cap U(A) = \emptyset$ . By the definition of  $U(A)$ ,  $U(A) + P = U(A)$ , and this yields  $\{y + tz \mid t > 0\} \cap U(A) = \emptyset$ . This means that  $y \in \partial U(A)$ .

*proof of Theorem 4.* Let  $x_0$  be an arbitrary point in  $U(A)$ . Since  $P$  is algebraically closed,  $P$  can not include any straight line. Indeed if  $\{x + ty \mid t \in \mathbb{R}\} \subset P$  for some  $y \neq 0$ , then  $\{ty \mid t \in \mathbb{R}\} \subset P \cup \partial P = P$  and this contradicts (P2). Hence for a positive element  $x \neq 0$ , there exists  $t_1 = \max\{t \geq 0 \mid x_0 - tx \in U(A)\}$ . If we put  $x_1 = x_0 - t_1 x$ , then  $x_1 \in \partial U(A)$  and it follows from Lemma 3 that  $U(A)_{x_1} \subset \partial U(A)$ . Since  $U(A)_{x_1}$  is a convex set and  $\text{int } U(A) \neq \emptyset$ , we can apply the separation theorem and there exists a hyper plane  $H$  which separates  $U(A)_{x_1}$  and  $U(A)$ .  $U(A)_{x_1} \subset (x_1 - P) \cap H$  and this is a straight half line by the assumption. Moreover, since  $U(A)$  can not include the whole straight line,  $U(A)_{x_1}$  is the form  $\{\lambda x_1 + (1 - \lambda)z \mid 0 \leq \lambda \leq 1\}$  where  $z \leq x_1$ . Clearly,  $z$  is a minimal element of  $U(A)$  and  $z \leq x_0$ , and this completes the proof. Q.E.D.

### §3 EXAMPLES

Let  $E$  be the space of all symmetric matrices of  $M_2(\mathbb{R})$ , and let  $P$  be the set of all positive semi definite matrices in  $E$ . Then  $(E, P)$  is m.o.c., but it is not a lattice.  $E$  and  $P$  can be identified with  $\mathbb{R}^3$  and

$$P = \{(x, y, z) \in \mathbb{R}^3 \mid z^2 \leq xy, 0 \leq x, 0 \leq y\}$$

respectively. It is easy to see that every exposed face of the positive cone  $P$  is 1-dimensional except the trivial face  $\{0\}$ , and  $P$  satisfies the condition in Theorem 3. Hence, by some simple calculations, we can determine the set  $a \vee b$  for incomparable pair  $a, b \in E$ .

Next we investigate the relation between the condition of Theorem 2 and that of Theorem 3. For a partially ordered linear space  $(E, P)$ , we say that the positive cone  $P$  satisfies condition  $(\mathfrak{F})$  when  $\dim C \leq 1$  for every  $C \in \mathfrak{F}(P)$ . In finite dimensional cases,  $P$  does not satisfy the condition  $(\mathfrak{F})$  when  $P$  is a closed convex cone generated by a finite set. On the other hand, such a positive cone satisfies monotone order completeness. This means that monotone order completeness does not imply the condition  $(\mathfrak{F})$ . Now we show an example in order to see the converse implication is also not true.

Let  $E$  be the linear space consisting of all sequences  $x = (x_1, x_2, \dots)$  ( $x_i \in \mathbb{R}$ ) such that  $x_i = 0$  except for finite number of  $i = 1, 2, \dots$ . We define

$$P = \{x = (x_1, x_2, \dots) \mid x_1 \geq \left(\sum_{i=2}^{\infty} x_i^2\right)^{\frac{1}{2}}\}.$$

Then it is easy to see that  $P$  is algebraically closed and  $\text{int } P \neq \emptyset$ . Indeed  $(1, 0, 0, \dots) \in \text{int } P$ . Let  $C \in \mathfrak{F}(P)$  and let  $x = (x_1, x_2, \dots), y = (y_1, y_2, \dots)$  be two points in  $C \setminus \{0\}$ . Since  $x, y \in \partial P$ ,  $x_1^2 = \sum_{i=2}^{\infty} x_i^2$ , and  $y_1^2 = \sum_{i=2}^{\infty} y_i^2$ . By the convexity of  $C$ , we also have  $\frac{1}{2}(x + y) \in \partial P$ , and hence  $(x_1 + y_1)^2 = \sum_{i=2}^{\infty} (x_i + y_i)^2$ . By simple calculation, we obtain  $x = \lambda y$  for some  $\lambda > 0$ . This means that  $\dim C = 1$ , and that  $P$  satisfies the condition  $(\mathfrak{F})$ . Thus Theorem 3 and Theorem 4 are applicable in this case.

We will show that  $(E, P)$  is not m.o.c. We define a sequence  $\{a_n\} \subset E$  by

$$a_n = \left(\frac{1}{2^n}, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots, \frac{1}{2^n}, 0, 0, \dots\right) \quad (n = 1, 2, \dots).$$

Then we have  $a_1 \geq a_2 \geq a_3 \geq \dots$ . Moreover, since  $(\frac{1}{2})^2 + (\frac{1}{4})^2 + (\frac{1}{8})^2 + \dots = \frac{1}{3}$ , we can see that  $(-\sqrt{\frac{1}{3}}, 0, 0, \dots)$  is a lower bound of  $\{a_n\}$ . Let  $b = (b_1, b_2, \dots, b_i, 0, 0, \dots)$

be an arbitrary lower bound of  $\{a_n\}$ . Then an element of the form  $c = (b_1 + \lambda, b_2, b_3, \dots, b_i, \mu, 0, 0, \dots)$  always satisfies  $b \not\preceq c$  when  $\lambda > 0$ . It is easy to see that we can choose  $\lambda$  and  $\mu$  such that  $c$  is also a lower bound of  $\{a_n\}$ . This means that the greatest lower bound of  $\{a_n\}$  does not exist, and  $(E, P)$  is not m.o.c..

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